Solution 8

1. Consider the problem of minimizing $f(x,y,z) = (x+1)^2 + y^2 + z^2$ subjecting to the constraint $g(x,y,z) = z^2 - x^2 - y^2 - 1,\ z > 0$. First solve it by eliminating z and then by Lagrange multipliers.

   Solution. Old method. From $g = 0$ get $z^2 = x^2 + y^2 + 1$. Plug in f to get $h(x,y) = (x+1)^2 + y^2 + x^2 + y^2 + 1$. When (x_0, y_0, z_0) is a local minimizer of f subject to $g = 0$, (x_0, y_0) is a local minimizer of $h(x,y)$. Hence $h_x = h_y = 0$ at (x_0, y_0) which yields

   $$2(x+1) + 2x = 0, \quad 2y + 2y = 0\ ,$$

   so $x = -1/2, y = 0$. We conclude that $(-1/2, 0, \sqrt{5}/2)$ is a critical point and hence a candidate for the local minimizer. (With further reasoning, it is really a global minimizer.)

   New method, there is some λ such that

   $$x + 1 = \lambda x,\ y = \lambda y,\ z = -\lambda z,\ z^2 - x^2 - y^2 = 1\ .$$

   The fourth equation implies that z is positive, so the third equation yields $\lambda = -1$. Then we get $x = -1/2, y = 0$ and $z = \sqrt{5}/2$.

   Note. Usually we don't have to check the condition $\nabla g \neq (0,0,0)$ before applying the theorem on Lagrange multipliers. You may check it if you like when everything is done.

2. Let $f, g_1, \cdots, g_m$ be C^1-functions defined in some open U in $\mathbb{R}^{n+m}$. Suppose (x_0, y_0) is a local extremum of f in $\{(x,y) \in U :\ g_1(x,y) = \cdots = g_m(x,y) = 0\ \}$. Assuming that $D_y G(x_0, y_0)$ is invertible where $G = (g_1, \cdots, g_m)$, show that there are $\lambda_1, \cdots, \lambda_m$ such that

   $$\nabla f + \lambda_1 \nabla g + \cdots + \lambda_m \nabla g_m = 0\ ,$$

   at (x_0, y_0).

   Solution. Similar to the special case $f(x,y,z)$ over $g(x,y,z) = 0$ in notes. What we need is a statement from linear algebra: Let E be an n-dimensional subspace of R^{n+m} and $u_1, \cdots, u_m$ are m-many independent vectors perpendicular to E. Then for any w perpendicular to E, $w + \sum_{j=1}^m \lambda_j u_j = 0$. Proof: Pick an orthonormal basis of E, $v_1, \cdots, v_n$. Then $v_1, \cdots, v_n, u_1, \cdots, u_m$ form a basis of $\mathbb{R}^{n+m}$. So

   $$w + \mu_1 v_1 + \cdots + \mu_n v_n + \lambda_1 u_1 + \cdots + \lambda_m u_m = 0\ .$$

   Taking inner product with v_k, we get $0 = w \cdot v_k + \mu_k = \mu_k$ for all k. Hence $w + \sum_{j=1}^m \lambda_j u_j = 0$.

3. Let $f \in C(R)$ where R is a closed rectangle. Suppose x solves $x' = f(t,x)$ for $t \in (a,b)$ with $(t, x(t)) \in R$. Show that x can be extended to be a solution in $[a,b]$.

   Solution. First, since $(t, x(t))$ remains in R which is bounded, there is $\{t_n\}, t_n \to b^-$ such that $x(t_n) \to z$ for some z. We claim in fact $x(t) \to z$ as $t \to b^-$. For $\varepsilon > 0$, take δ to satisfy $\delta < \varepsilon/(2M), M = \sup_R |f|$. Then for $t, b - t < \delta$, we can find some $t_n \in (t,b)$ such that $|x(t_n) - z| < \varepsilon/2$. Then

   $$\begin{aligned} |x(t) - z| &\leq |x(t) - x(t_n)| + |x(t_n) - z| \\ &< \left| \int_{t_n}^t f(s, x(s))\, ds \right| + \frac{\varepsilon}{2} \\ &\leq M|t_n - t| + \frac{\varepsilon}{2} \\ &\leq \varepsilon\ . \end{aligned}$$

By defining $x(b) = z$, we see that $x(t)$ is continuous on $(a, b]$. In the relation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \quad t \in (a, b) \ ,$$

we can let $t \to b^-$ to show it remains true in $(a, b]$. It follows that $x' = f(t, x)$ at $t = b$ where x' is understood as the left derivative. Similarly, we can show the solution extends to $[a, b)$ too.

4. Let $f \in C(R)$ where R is a closed rectangle satisfy a Lipschitz condition in R. Suppose that x solves $x' = f(t, x)$ for $t \in [a, b]$ where $(b, x(b))$ lies in the interior of R. Show that there is some $\delta > 0$ such that x can be extended as a solution in $[a, b + \delta]$.

   Solution. Solve the IVP of the equation passing the point $(b, x(b))$. Since this point lies in the interior of R, we can find a small rectangle R_1 inside R taking this point as the center. By applying the Picard-Lindelof theorem to R_1 we obtain a solution extending beyond b, that is, in $(b - \delta, b + \delta)$ for some δ. By uniqueness (see Proposition 3.14) it coincides with the old solution in their common interval of existence, hence the solution exists on $[a, b + \delta)$. (By Question 4 actually in $[a, b + \delta]$, but that is not essential.)

5. Let $D = (a, b) \times \mathbb{R}$ and $f \in C(\overline{D})$ satisfy a Lipschitz condition. Let x be a maximal solution to the (IVP) $x' = f(t, x),\ x(t_0) = x_0,\ t_0 \in (a, b)$ over the maximal interval (α, β). Show that if $\beta < b$, $x(t) \to \infty$ or $x(t) \to -\infty$ as $t \uparrow \beta$.

   Solution Assume $\beta < b \leq \infty$ and in particular β is finite. In case $x(t)$ does not tend to ∞ or $-\infty$, we can find some $t_n, t_n \uparrow \beta$ such that $x(t_n) \to z$ for some z. (see below) As $(\beta, z) \in D$ we can find a small rectangle R of the form $[\beta - \delta, \beta + \delta] \times [z - \rho, z + \rho]$ in D. For large n, the rectangle $[t_n - \delta/2, t_n + \delta/2] \times [x(t_n) - \rho/2, x(t_n) + \rho/2]$ is contained in R. Now solve the (IVP) starting at $(t_n, x(t_n))$. According to the Picard-Lindelof theorem, the solution exists over some interval $(t_n - a', t_n + a')$ where a' depends only on δ, ρ, M and L. (M is the supremum of $|f|$ over R.) As $t_n \uparrow \beta, t_n + a' > \beta$ for sufficiently large n. That is, the solution exists in beyond β, contradiction holds.

   When $x(t)$ does not blow up at $\pm\infty$, there are three possibilities (a) $\sup_t x(t) = \infty$, (b) $\inf_t x(t) = -\infty$, and (c) $x(t)$ is bounded. Let us consider (a). As $x(t)$ does not blow up to ∞ at β, there exist some M and $\tau_n \to \beta$ such that $x(\tau_n) < M$. On the other hand, as $\sup_t x(t) = \infty$, there is $s_n \to \beta$ such that $x(s_n) > M$. By the continuity of $x(t)$, there is some $t_n \to \beta$ such that $x(t_n) = M$ for all n. We may take $z = M$ in this case. (b) can be treated similarly and (c) is evident.

6. Let f and g be two continuous functions in $\overline{D}$ both satisfying a Lipschitz condition and $f < g$ everywhere. Let x and y be the respective solutions to the (IVP) of f and g satisfying $x(t_0) < y(t_0)$. Show that $x(t) < y(t), t \geq t_0$, as long as they exist.

   Solution By continuity, $x(t) < y(t)$ for small $t > t_0$. Let t_1 be the first time the two solution curves touch, that is, $x(t_1) = y(t_1)$ and $x(t) < y(t), t < t_1$. Then for $t \in (t_0, t_1)$,

$$\frac{x(t_1) - x(t)}{t_1 - t} > \frac{y(t_1) - y(t)}{t_1 - t} \ .$$

   Letting $t \uparrow t_1$, $x'(t_1) \geq y'(t_1)$, contradicting $x' = f < g = y'$. Hence $x(t) < y(t), t > t_0$, as long as they exist.

7. Let $D = \mathbb{R}^2$ and $f \in C(\mathbb{R}^2)$ satisfy a Lipschitz condition. Suppose that $|f(t,x)| \le C(1+|x|)$ everywhere. Show that all maximal solutions exists on $(-\infty, \infty)$. Hint: Use the previous two questions.

   Solution Let $x(t)$ be the maximal solution of the (IVP) to $x' = f(t,x), x(t_0) = x_0$. Solve the (IVP) $y' = C(1+|y|)+1,\ y(t_0) = x_0 + k$ and $z' = -C(1+|z|) - 1, z(t_0) = x_0 - k$ to get the maximal solutions $y(t)$ and $z(t)$ which exist for all time. Here k is a large number so that $x_0 + k > 0$ and $x_0 - k < 0$. Therefore, y is positive and z is negative and they solve the linear equations $y' = C(1+y)+1$ and $z' = -C(1-z) - 1$ respectively. We know that solutions to linear equations exist for all time, see Example 3.14. By the previous question, $z(t) < x(t) < y(t)$ as long as x exists. As y, z exists for all t, in any finite interval $[t_0, t]$, x cannot blow up to $\pm\infty$. By Question 5, x must exist on $[t_0, \infty)$. A similar argument shows that it also exists on $(-\infty, t_0]$.

8. Provide a proof to Theorem 3.15 (Picard-Lindelof theorem for systems).

   Solution The proof is basically the same as in the equation case. Tutor will do it in class.